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Relative graded Clifford theory

C. Năstăsescu

Facultatea de Mathematică, Str. Academiei 14, R. 70109 Bucharest 1, Romania

B. Torrecillas*

Departamento de Algebra, Universidad de Granada, Campus de Almeria, 04071 Almeria, Spain

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Abstract

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We give a relative version of the 'Graded Clifford Theorem'. The relative graded Clifford theorem is a powerful tool in the study of $\mathscr{C}$-cocritical objects of the category R-**gr** where $\mathscr{C}$ is a rigid localizing subcategory of R-**gr**. We apply the result to the study of Gabriel (Krull) dimension of a graded module.

1. Introduction

Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a G-graded ring where G is a group with identity element 1 and let R-**gr** be the category of G-graded left R-modules. If Σ is a gr-simple module, i.e. a simple object of the Grothendieck category R-**gr**, then $\sigma_R[\Sigma]$ denotes the full additive subcategory of R-**mod** whose objects are the left R-modules subgenerated by Σ (i.e. isomorphic to submodules of quotient modules of direct sums of copies of Σ). Then the 'graded Clifford theorem' asserts that $\sigma_R[\Sigma]$ is equivalent to the category Δ-**mod** of left Δ-modules, where $\Delta = \mathrm{End}_R(\Sigma)$. The equivalence is given by the functors

$$\mathrm{Hom}_R(\Sigma, -) : \sigma_R[\Sigma] \to \Delta\text{-}\mathbf{mod}$$

and

Correspondence to: Professor B. Torrecillas, Departamento de Algebra, Universidad de Granada, Campus de Almeria, 04071 Almeria, Spain.
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$$\Sigma \otimes_{\Delta} - : \Delta\textbf{-mod} \to \sigma_R[\Sigma] .$$

This result was first proved by Dade [2, 3] and it was extended to a more general context in [5], where it was also shown that the graded Clifford theorem is a very powerful tool in the study of gr-simple modules.

The aim of this paper is to give a more general version of the graded Clifford theorem.

More exactly, if $\mathscr{C}$ is a rigid localizing subcategory of category R-**gr** and $M \in R$-**gr**, then M is called $\mathscr{C}$-simple (or $\mathscr{C}$-cocritical) if (1) M is $\mathscr{C}$-torsionfree and (2) for any nonzero graded submodule M' of M, $M/M' \in \mathscr{C}$. It is clear that if $\mathscr{C} = \{0\}$, then M is $\mathscr{C}$-simple if and only if M is gr-simple.

If $\mathscr{C}$ is a rigid localizing subcategory of R-**gr**, we denote by $\bar{\mathscr{C}}$ the smallest localizing subcategory of R-**mod** containing $\mathscr{C}$. We denote by $\sigma_R[M]/\bar{\mathscr{C}} \cap \sigma_R[M]$ the quotient category of $\sigma_R[M]$ by $\bar{\mathscr{C}} \cap \sigma_R[M]$, and by

$$T : \sigma_R[M] \to \sigma_R[M]/\bar{\mathscr{C}} \cap \sigma_R[M]$$

the canonical functor. We can consider the object $\Sigma = T(M)$ and its ring of endomorphisms $\Delta = \mathrm{End}(\Sigma)$. Then the main result of this paper is Theorem 5.2 (the Relative Clifford Theorem) which says that if M is $\mathscr{C}$-simple then the quotient category $\sigma_R[\Sigma]/\bar{\mathscr{C}} \cap \sigma_R[\Sigma]$ is equivalent to the category Δ-**mod** via the functor

$$\mathrm{Hom}_{\sigma_R[M]/\bar{\mathscr{C}} \cap \sigma_R[M]}(M, -) : \sigma_R[M]/\bar{\mathscr{C}} \cap \sigma_R[M] \to \Delta\textbf{-mod} .$$

The structure of the ring Δ, given by Theorem 5.4, is very important in the applications. The applications of the Relative Clifford Theorem to the study of Gabriel (Krull) dimension are given in Section 6 and we also obtain a relative Maschke's Theorem.

2. Notation and preliminaries

Throughout this paper, all rings R will be associative and with identity, and all modules will be left R-modules. The category of left R-modules will be denoted by R-**mod**. Let G be a (multiplicative) group with identity, together with a direct-sum decomposition $R = \bigoplus_{\sigma \in G} R_\sigma$ (as additive subgroups) such that

$$R_\sigma R_\tau \subseteq R_{\sigma\tau} \quad \text{for all } \sigma, \tau \in G . \tag{1}$$

It is well known that R_1 is a subring of R, and $1 \in R_1$. If in (1) we have equality, i.e. $R_\sigma R_\tau = R_{\sigma\tau}$ for all $\sigma, \tau \in G$, then R is called a *strongly graded* ring. It is easy to see that R is strongly graded if and only if $R_\sigma R_{\sigma^{-1}} = R_1$ for any $\sigma \in G$. If for any $\sigma \in G$, R_σ contains an invertible element, then R is called a *crossed product*. It is obvious that if R is a crossed product, then R is strongly graded. By a left G-graded R-module we understand a left R-module M plus an internal direct-sum decomposition $M = \bigoplus_{\sigma \in G} M_\sigma$ (as additive subgroups) such that

$$R_\sigma M_\tau \subseteq M_{\sigma\tau} \quad \text{for all } \sigma,\tau \in G .$$

We denote by R-**gr** the category of left G-graded R-modules. If $M = \bigoplus_{\sigma \in G} M_\sigma$ and $N = \bigoplus_{\sigma \in G} N_\sigma$ are two G-graded modules, then $\mathrm{Hom}_{R\text{-}\mathbf{gr}}(M, N)$ consists of the R-homomorphisms $f : M \to N$ such that $f(M_\sigma) \subseteq N_\sigma$ for every $\sigma \in G$. As it is well known [11], R-**gr** is a Grothendieck category. If $M = \bigoplus_{\sigma \in G} M_\sigma$ is a graded R-module, $h(M)$ will stand for the set of all homogeneous elements of M, i.e. $h(M) = \bigcup_{\sigma \in G} M_\sigma \setminus \{0\}$. If $m \in M$, $m \neq 0$ we can write $m = \sum_{\sigma \in G} m_\sigma$ where $m_\sigma \in M_\sigma$; the finite set $\{m_\sigma \mid \sigma \in G,\ m_\sigma \neq 0\}$ is called the set of homogeneous components of m. If $M = \bigoplus_{\lambda \in G} M_\lambda$ is a graded R-module and $\sigma \in G$, then the σ-suspension of M is defined as the graded module $M(\sigma)$ obtained from M, by setting $M(\sigma)_\lambda = M_{\lambda\sigma}$. The σ-suspension functor

$$T_\sigma : R\text{-}\mathbf{gr} \to R\text{-}\mathbf{gr}$$

defined by $T_\sigma(M) = M(\sigma)$ is an equivalence of categories. We denote by $G\{M\} = \{\sigma \in G \mid M(\sigma) \simeq M\}$. Clearly $G\{M\}$ is a subgroup of G which is called the *stabilizer* of M. If $G\{M\} = G$, then M is called *G-invariant*. It is clear that for any $M \in R$-**gr**, the graded module $\bigoplus_{\sigma \in G} M(\sigma)$ is G-invariant. Let M and N be graded R-modules. For each $\sigma \in G$ we set

$$\begin{aligned} &\mathrm{HOM}_R(M, N)_\sigma \\ &\quad = \{ f : M \to N \mid f \text{ is } R\text{-linear and } f(M_\lambda) \subseteq N_{\lambda\sigma}\ \forall \lambda \in G\} \\ &\quad = \mathrm{Hom}_{R\text{-}\mathbf{gr}}(M, N(\sigma)) = \mathrm{Hom}_{R\text{-}\mathbf{gr}}(M(\sigma^{-1}), N) . \end{aligned}$$

$\mathrm{HOM}_R(M, N)_\sigma$ is an additive subgroup of the group $\mathrm{Hom}_R(M, N)$ of all R-homomorphisms from M to N, and $\mathrm{HOM}_R(M\ N) = \bigoplus_{\sigma \in G} \mathrm{HOM}_R(M, N)_\sigma$ is a subgroup of $\mathrm{Hom}_R(M, N)$ and it is, in fact, a G-graded abelian group. Clearly, $\mathrm{HOM}_R(M, N)_1$ is just $\mathrm{Hom}_{R\text{-}\mathbf{gr}}(M, N)$. It is well known that if M is finitely generated or G is a finite group, then $\mathrm{HOM}_R(M, N) = \mathrm{Hom}_R(M, N)$ [11]. If $N = M$, we denote $\mathrm{END}_R(M) = \mathrm{HOM}_R(M, M)$; then $\Lambda = \mathrm{END}_R(M)$ is a G-graded subring of $\Delta = \mathrm{End}_R(M)$.

Let $\mathscr{A}$ be a Grothendieck category. A full subcategory $\mathscr{C}$ of $\mathscr{A}$ is called *closed* (see [4, p. 395]) if $\mathscr{C}$ is closed under subobjects, quotient objects and direct sums. If $\mathscr{C}$ is, furthermore, closed under extensions, then $\mathscr{C}$ is called a *localizing subcategory* of $\mathscr{A}$. It may be easily seen that a closed subcategory of a Grothendieck category is also a Grothendieck category. If $\mathscr{C}$ is closed, the sum of all the subobjects, $t_{\mathcal{A}}(M)$, of $M \in \mathcal{A}$ which belong to $\mathcal{C}$, defines a left exact subfunctor $t_{\mathcal{C}} : \mathcal{A} \to \mathcal{A}$ of the identity of $\mathcal{A}$, which is called the preradical functor associated to $\mathcal{C}$. If $M \in \mathcal{A}$ and $M = t_{\mathcal{C}}(M)$, then M is said to be *$\mathcal{C}$-torsion*; if $t_{\mathcal{C}}(M) = 0$, then M is called a *$\mathcal{C}$-torsionfree* object. If $\mathcal{C}$ is a Grothendieck category and $M \in \mathcal{A}$ is an arbitrary object, we denote by $\sigma_{\mathcal{A}}[M]$ (or shortly $\sigma[M]$) the class of all the objects of $\mathcal{A}$ subgenerated by M (i.e. isomorphic to subobjects of quotient objects of direct sums of copies of M). Then $\sigma_{\mathcal{A}}[M]$ is a closed subcategory of $\mathcal{A}$ containing M. As in [15, p. 122], an arbitrary Grothendicek category $\mathcal{A}$ is a locally finitely generated (resp. locally noetherian, locally artinian) category if it has a family of finitely generated (resp. noetherian, artinian) generators. (Recall that an object $M \in \mathcal{A}$ is called *finitely generated* if, whenever $M = \sum_{i \in I} M_i$ for a direct family of subobjects M_i of M, there exists an index i_0 such that $M = M_{i_0}$.)

Proposition 2.1. *Assume that $\mathcal{A}$ is a locally finitely generated Grothendieck category.*

(i) *If $M \in \mathcal{A}$, then $\sigma_{\mathcal{A}}[M]$ is locally finitely generated.*

(ii) *If M is noetherian (resp. artinian), then $\sigma_{\mathcal{A}}[M]$ is locally noetherian (resp. locally artinian).*

Proof. (i) In fact, we have the more general result: if $\mathcal{C}$ is a closed subcategory of $\mathcal{A}$, then $\mathcal{C}$ is locally finitely generated. Indeed, assume that $\{U_i \mid i \in I\}$ is a family of finitely generated generators. It is clear that if we denote by $\{V_j \mid j \in J\}$ the family of all objects V_j from $\mathcal{C}$ that are homomorphic images of some object U_i, it is a family of generators for the category $\mathcal{C}$. Since U_i is finitely generated, any homomorphic image of it is also finitely generated.

(ii) With the above notation, we consider the particular case $\mathcal{C} = \sigma_{\mathcal{A}}[M]$. Since $V_j \in \mathcal{C}$, there exists an epimorphism

$$M^{(\Lambda)} \xrightarrow{u} X \to 0$$

such that $V_j \subseteq X$ (V_j is a subobject of X). Since V_j is finitely generated, there exists a finite subset K of Λ such that V_j is a subobject of $X' = u(M^{(K)})$. If M is noetherian (resp. artinian), then $M^{(K)}$ is noetherian (resp. artinian) and therefore X' is noetherian (resp. artinian). Hence V_j is noetherian (resp. artinian). Thus $\sigma_{\mathcal{A}}[M]$ is locally noetherian (resp. locally artinian). $\square$

Assume now that $\mathcal{C}$ is a localizing subcategory of $\mathcal{A}$. As in [4], $M \in \mathcal{A}$ is called *$\mathcal{C}$-closed* if $t_{\mathcal{C}}(M) = 0$ (i.e. M is $\mathcal{C}$-torsionfree) and for any exact sequence

$$\begin{tikzcd} 0 \arrow[r] & N' \arrow[r, "u"] \arrow[d, "f"'] & N \arrow[r] \arrow[dl, "g"] & \operatorname{Coker} u \arrow[r] & 0 \\ & M & & & \end{tikzcd}$$

such that Coker $u \in \mathscr{C}$ and for any morphism $f : N' \to M$ there exists (a unique) morphism $g : N \to M$ such that $gu = f$.

If $\mathscr{C}$ is a localizing subcategory of $\mathscr{A}$, we can define the quotient category $\mathscr{A}/\mathscr{C}$, which is also a Grothendieck category. We denote $T_{\mathscr{C}} : \mathscr{A} \to \mathscr{A}/\mathscr{C}$, and $S_{\mathscr{C}} : \mathscr{A}/\mathscr{C} \to \mathscr{A}$ the canonical functors (see [4, Chapter III]). It is well known that $T_{\mathscr{C}}$ is an exact functor and $S_{\mathscr{C}}$ is a right adjoint for $T_{\mathscr{C}}$. Moreover, $S_{\mathscr{C}}$ is a left exact functor. If $\phi : T_{\mathscr{C}}S_{\mathscr{C}} \to 1_{\mathscr{A}/\mathscr{C}}$ and $\psi : 1_{\mathscr{A}} \to S_{\mathscr{C}}T_{\mathscr{C}}$ are the natural transformations of functors, then ϕ is an isomorphism. Further, if $M \in \mathscr{A}$ then we have the exact sequence

$$0 \to \operatorname{Ker} \psi(M) \to M \xrightarrow{\psi(M)} S_{\mathscr{C}}T_{\mathscr{C}}(M) \to \operatorname{Coker} \psi(M) \to 0\,,$$

where Ker $\psi(M)$ and Coker $\psi(M)$ belongs to $\mathscr{C}$. Also $M \in \mathscr{A}$ is $\mathscr{C}$-closed if and only if the canonical morphism $\psi(M)$ is an isomorphism.

3. Adjoint functors and localization

Recall that if $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ are additive functors between Grothendieck categories, F is a left adjoint of G (or G is a right adjoint of F) if there exists a natural equivalence

$$\varphi : \operatorname{Hom}_{\mathscr{B}}(F(-), -) \to \operatorname{Hom}_{\mathscr{A}}(-, G(-))\,.$$

It is well known that in this case F is right exact and G is left exact.

Theorem 3.1. *Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be functors between Grothendieck categories such that F is left adjoint of G. Let $\mathscr{C}$ (resp. $\mathscr{D}$) be a localizing subcategory of $\mathscr{A}$ (resp. $\mathscr{B}$). Assume that $F(\mathscr{C}) \subseteq \mathscr{D}$. Then the following assertions hold*:

(i) *If $N \in \mathscr{B}$ is $\mathscr{D}$-torsionfree, then $G(N)$ is $\mathscr{C}$-torsionfree.*

(ii) *If F is an exact functor and $N \in \mathscr{B}$ is $\mathscr{D}$-closed, then $G(N)$ is $\mathscr{C}$-closed.*

Proof. (i) Let $X = T_{\mathscr{C}}(G(N))$. We have

$$\operatorname{Hom}_{\mathscr{A}}(X, G(N)) \simeq \operatorname{Hom}_{\mathscr{B}}(F(X), N)\,.$$

Since $F(\mathscr{C}) \subseteq \mathscr{D}$, then $F(X) \in \mathscr{D}$, and therefore $\operatorname{Hom}_{\mathscr{B}}(F(X), N) = 0$; hence $\operatorname{Hom}_{\mathscr{A}}(X, G(N)) = 0$. Thus $X = 0$.

(ii) We consider the exact sequence in the category $\mathscr{A}$

$$0 \to X' \xrightarrow{u} X \to \operatorname{Coker} u \to 0$$

where $\operatorname{Coker} u \in \mathscr{C}$. Since F is exact, then we have an exact sequence

$$0 \to F(X') \xrightarrow{F(u)} F(X) \to F(\operatorname{Coker} u) \to 0$$

where $F(\operatorname{Coker} u) \in \mathscr{D}$. Since N is $\mathscr{D}$-closed, we have the exact sequence

$$\operatorname{Hom}_{\mathscr{B}}(F(X), N) \to \operatorname{Hom}_{\mathscr{B}}(F(X'), N) \to 0 .$$

Applying the natural transformation φ, we get that the sequence

$$\operatorname{Hom}_{\mathscr{A}}(X, G(N)) \to \operatorname{Hom}_{\mathscr{A}}(X', G(N)) \to 0$$

is exact, and therefore $G(N)$ is $\mathscr{C}$-closed. □

Theorem 3.2. *With the hypothesis of Theorem* 3.1, *denote by*

$$\mathscr{A} \underset{S_1}{\overset{T_1}{\rightleftarrows}} \mathscr{A}/\mathscr{C} \quad \text{and} \quad \mathscr{B} \underset{S_2}{\overset{T_2}{\rightleftarrows}} \mathscr{B}/\mathscr{D}$$

the canonical functors. Assume that F is an exact functor. If $\bar{F} = T_2 \circ F \circ S_1$ and $\bar{G} = T_1 \circ G \circ S_2$, then $\bar{F}$ is a left adjoint of the functor $\bar{G}$, $\bar{F}$ is exact and $\bar{F} \circ T_1 \simeq T_2 \circ F$. Moreover, if $G(\mathscr{D}) \subseteq \mathscr{C}$ and G is exact, then $\bar{G}$ is exact.

Proof. Let $\mathscr{Q}$ denote the full subcategory of $\mathscr{A}$ consisting of the $\mathscr{C}$-closed objects, then T_1 and S_1 induce an equivalence between $\mathscr{Q}$ and $\mathscr{A}/\mathscr{C}$. By Theorem 3.1, $GS_2(\mathscr{B}/\mathscr{D}) \subseteq \mathscr{Q}$, hence we can regard $G \circ S_2$ as a functor from $\mathscr{B}/\mathscr{D}$ to $\mathscr{Q}$. By composition we obtain that $\bar{F}$ is a left adjoint of $\bar{G}$.

Now, we prove that $\bar{F}$ is exact. Since $\bar{F}$ is a left adjoint, it is enough to see that it is left exact. Now S_1 is left exact, and F and T_2 are exact, the result follows.

We still need to prove that $\bar{F} \circ T_1 \simeq T_2 \circ F$. Indeed if $X \in \mathscr{A}$ then $(\bar{F} \circ T_1)(X) = (T_2 \circ F \circ S_1 \circ T_1)(X)$. But from the exact sequence

$$0 \to \operatorname{Ker} \psi_1(X) \to X \xrightarrow{\psi_1(X)} (S_1 \circ T_1)(X) \to \operatorname{Coker} \psi_1(X) \to 0$$

where $\operatorname{Ker} \psi_1$ and $\operatorname{Coker} \psi_1$ belong to $\mathscr{C}$ (here $\psi_1(X)$ is the canonical morphism), we obtain the exact sequence

$$\begin{aligned} 0 &\to (T_2 \circ F)(\operatorname{Ker} \psi_1(X)) \to (T_2 \circ F)(X) \xrightarrow{(T_2 \circ F)(\psi_1(X))} \\ &\to (T_2 \circ F)(S_1 \circ T_1)(X) \to (T_2 \circ F)(\operatorname{Coker} \psi_1(X)) \to 0 . \end{aligned}$$

Since $F(\mathscr{C}) \subseteq \mathscr{D}$ and $T_2(\mathscr{D}) = \{0\}$, then $(T_2 \circ F)(\psi_1(X))$ is an isomorphism. Hence $\bar{F} \circ T_1 \simeq T_2 \circ F$. □

4. Localization in R-gr

Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a G-graded ring. We denote by $U : R\text{-}\mathbf{gr} \to R\text{-}\mathbf{mod}$ the forgetful functor; U is an exact functor. It is well known that U has a right adjoint [11] $F : R\text{-}\mathbf{mod} \to R\text{-}\mathbf{gr}$, which is defined as follows: if $M \in R\text{-}\mathbf{gr}$, then $F(M)$ is the additive group $\bigoplus_{\sigma \in G} ({}^{\sigma}M)$ (where each ${}^{\sigma}M$ is a copy of M, ${}^{\sigma}M = \{{}^{\sigma}x \mid x \in M\}$) with the R-module structure given by $a * {}^{\sigma}x = {}^{\tau\sigma}(ax)$ for $a \in R_\tau$. Obviously, the gradation of $F(M)$ is given by $F(M)_\sigma = {}^{\sigma}M$, $\sigma \in G$, and if $f \in \mathrm{Hom}_R(M, N)$, then $F(f) \in \mathrm{Hom}_{R\text{-}\mathbf{gr}}(F(M), F(N))$ is given by $F(f)({}^{\sigma}x) = {}^{\sigma}f(x)$. We remark that F is an exact functor. Note also that $U(F(M))$ need not be a direct sum of copies of M, since the component ${}^{\sigma}M$ is not an R-submodule, but just an R_1-submodule of $F(M)$. On the other hand, it is easy to see that if $M \in R\text{-}\mathbf{gr}$, then $F(U(M)) = \bigoplus_{\sigma \in G} M(\sigma)$ (see [13, Lemma 3.1]). If $M \in R\text{-}\mathbf{mod}$, we have the canonical epimorphism in R-**mod**

$$F(M) \xrightarrow{\alpha} M \to 0$$

such that $\alpha({}^{\sigma}x) = x$, $x \in M$.

Proposition 4.1. *The functor F commutes with direct sums.*

Proof. Let $M = \bigoplus_{i \in I} M_i$. Since $M_i \subseteq M$ we have the canonical morphism $u : \bigoplus_{i \in I} F(M_i) \to (M)$. We define the canonical morphism $v : F(M) \to \bigoplus_{i \in I} M_i$ in the following way: if ${}^{\sigma}x \in F(M)$, where $x = \sum_{i \in I} x_i$, $x_i \in M_i$, then we put $v({}^{\sigma}x) = \sum_{i \in I} {}^{\sigma}x_i$. It is easy to show that $v \circ u = \mathbf{1}_{\oplus_{i \in I} F(M_i)}$ and $u \circ v = \mathbf{1}_{F(M)}$. $\square$

Let now $\mathscr{C}$ be a closed subcategory of R-**gr**. If for any $M \in \mathscr{C}$ and $\sigma \in G$ we have $M(\sigma) \in \mathscr{C}$, then $\mathscr{C}$ is called a *rigid* closed subcategory of R-**gr**. We denote by $\bar{\mathscr{C}}$ the smallest closed subcategory of R-**mod** containing $\mathscr{C}$. By Proposition 2.1 of [5], an R-module M belongs to $\bar{\mathscr{C}}$ if and only if there exists $N \in \mathscr{C}$ such that M is isomorphic to a quotient of N. By Proposition 2.3 of [5], if $M \in \bar{\mathscr{C}}$, then $F(M) \in \mathscr{C}$.

The following result will be very useful in the sequel (it is another version of Propositions 2.1 and 2.3 of [5]):

Proposition 4.2. *With the above notation, we have the equality*

$$\bar{\mathscr{C}} = \{M \in R\text{-}\mathbf{mod} \mid F(M) \in \mathscr{C}\}\,.$$

Proof. Let $\mathscr{U} = \{M \in R\text{-mod} \mid F(M) \in \mathscr{C}\}$. Since F is an exact functor and commutes with arbitrary direct sums (Proposition 4.1) it follows that $\mathscr{U}$ is a closed subcategory of R-**mod**. If $M \in \mathscr{C}$, since $F(M) = \bigoplus_{\sigma \in G} M(\sigma)$ and since $\mathscr{C}$ is rigid, it follows that $F(M) \in \mathscr{C}$. Therefore $\mathscr{C} \subseteq \mathscr{U}$. Let $\mathscr{D}$ be another closed subcategory of R-**mod** such that $\mathscr{C} \subseteq \mathscr{D}$. If $M \in \mathscr{U}$, $F(M) \in \mathscr{C}$, and hence $F(M) \in \mathscr{D}$. Since M is a homomorphic image of $F(M)$ in R-**mod**, it follows that $M \in \mathscr{D}$. Hence $\mathscr{U} \subseteq \mathscr{D}$, and therefore $\mathscr{U} = \bar{\mathscr{C}}$. □

Proposition 4.3. *Let $\mathscr{A}$ and $\mathscr{C}$ be two rigid closed subcategories of R-**gr** such that $\mathscr{C} \subseteq \mathscr{A}$. If $\mathscr{C}$ is localizing subcategory in $\mathscr{A}$, then $\bar{\mathscr{C}}$ is a localizing subcategory in $\bar{\mathscr{A}}$.*

Proof. Since $\bar{\mathscr{C}}$ is a closed subcategory in R-**mod**, then $\bar{\mathscr{C}}$ is also a closed subcategory in $\bar{\mathscr{A}}$. We show that if we have the exact sequence

$$0 \to M' \to M \to M'' \to 0$$

such that $M', M'' \in \bar{\mathscr{C}}$ and $M \in \bar{\mathscr{A}}$, then $M \in \bar{\mathscr{C}}$.

Since F is an exact functor, we have the exact sequence

$$0 \to F(M') \to F(M) \to F(M'') \to 0\,.$$

By Proposition 4.2 we have that $F(M'), F(M'') \in \mathscr{C}$ and $F(M) \in \mathscr{A}$. Since $\mathscr{C}$ is localizing subcategory in $\mathscr{A}$, then $F(M) \in \mathscr{C}$. Since M is a homomorphic image of $F(M)$, it follows that $M \in \bar{\mathscr{C}}$. Thus $\bar{\mathscr{C}}$ is a localizing subcategory of $\bar{\mathscr{A}}$. □

Let $\mathscr{A}$ and $\mathscr{C}$ be two rigid closed subcategories of R-**gr** such that $\mathscr{C} \subseteq \mathscr{A}$ and $\mathscr{C}$ is a localizing subcategory of $\mathscr{A}$. We let $\bar{\mathscr{C}}$ and $\bar{\mathscr{A}}$ denote, as above, the smallest closed subcategories of R-**mod** containing $\mathscr{C}$ and $\mathscr{A}$, respectively. By Proposition 4.3 we have that $\bar{\mathscr{C}}$ is a localizing subcategory of $\bar{\mathscr{A}}$. We consider the functors

$$R\text{-}\mathbf{gr} \underset{F}{\overset{U}{\rightleftarrows}} R\text{-}\mathbf{mod}\,, \tag{2}$$

where U is the forgetful and F is the right adjoint of U. By Proposition 4.2, we have $F(\bar{A}) \subseteq \mathscr{A}$ and $F(\bar{\mathscr{C}}) \subseteq \mathscr{C}$. Hence we have the canonical functors

$$\mathscr{A} \underset{F}{\overset{U}{\rightleftarrows}} \bar{\mathscr{A}}\,. \tag{3}$$

In fact, in (3) U and F are the restrictions of the functors from (2). Let

$$\mathscr{A} \underset{T_1}{\overset{S_1}{\leftrightarrows}} \mathscr{A}/\mathscr{C} \quad \text{and} \quad \bar{\mathscr{A}} \underset{T_2}{\overset{S_2}{\leftrightarrows}} \bar{\mathscr{A}}/\bar{\mathscr{C}}$$

be the canonical functors associated to the quotient categories. We denote by

$$\bar{U} : \mathscr{A}/\mathscr{C} \to \bar{\mathscr{A}}/\bar{\mathscr{C}}\,,$$

$\bar{U} = T_2 \circ U \circ S_1$ and by

$$\bar{F} : \bar{\mathcal{A}}/\bar{\mathcal{C}} \to \mathcal{A}/\mathcal{C} ,$$

$\bar{F} = T_1 \circ F \circ S_2$. By Theorem 3.1, $\bar{F}$ is a right adjoint of $\bar{U}$ and $\bar{F}$ and $\bar{U}$ are exact functors. Also $(\bar{U}) \circ T_1 \simeq T_2 \circ U$. We give now some more properties of the functors $\bar{U}$ and $\bar{F}$.

Proposition 4.4. *The functor $\bar{F}$ commutes with direct sums.*

Proof. Let $(Y_i)_{i \in I}$ a family of objects from $\bar{\mathcal{A}}/\bar{\mathcal{C}}$ and put $Y = \bigoplus_{i \in I} Y_i$. We have the canonical morphism

$$\bigoplus_{i \in I} S_2(Y_i) \xrightarrow{\alpha} S_2(Y) .$$

Since T_2 is an exact functor that commutes with direct sums and $T_2 \circ S_2 \simeq 1_{\bar{\mathcal{A}}/\bar{\mathcal{C}}}$, then $T_2(\alpha)$ is an isomorphism. Then $\operatorname{Ker} \alpha$ and $\operatorname{Coker} \alpha$ belong to $\bar{\mathcal{C}}$. Hence $F(\operatorname{Ker} \alpha), F(\operatorname{Coker} \alpha) \in \mathcal{C}$ and we get that $(T_1 \circ F)(\alpha)$ is an isomorphism. Since F commutes with direct sums (Proposition 4.1), we have

$$(T_1 \circ F)\Big(\bigoplus_{i \in I} S_2(Y_i)\Big) \simeq \bigoplus_{i \in I} (T_1 \circ F \circ S_2)(Y_i)$$

and therefore $\bar{F}(Y) \simeq \bigoplus_{i \in I} \bar{F}(Y_i)$. □

Recall [8, 14], that in a Grothendieck category $\mathcal{A}$, an object $\Sigma \in \mathcal{A}$ is called *small* if the functor

$$\operatorname{Hom}_{\mathcal{A}}(\Sigma, -) : \mathcal{A} \to \mathbf{Ab}$$

commutes with direct sums. Obviously, any finitely generated object in $\mathcal{A}$ is small.

Proposition 4.5. *With the above notation, if $\Sigma \in \mathcal{A}/\mathcal{C}$ is a small object, then $\bar{U}(\Sigma)$ is small in $\bar{\mathcal{A}}/\bar{\mathcal{C}}$.*

Proof. Let $Y = \bigoplus_{i \in I} Y_i$ be a direct sum in $\bar{\mathcal{A}}/\bar{\mathcal{C}}$. Then by Theorem 3.2 and Proposition 4.4 we have that

$$\begin{aligned} &\operatorname{Hom}_{\bar{\mathcal{A}}/\bar{\mathcal{C}}}\Big(\bar{U}(\Sigma), \bigoplus_{i \in I} Y_i\Big) \\ &\quad \simeq \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}\Big(\Sigma, \bar{F}\Big(\bigoplus_{i \in I} Y_i\Big)\Big) \simeq \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}\Big(\Sigma, \bigoplus_{i \in I} \bar{F}(Y_i)\Big) \\ &\quad \simeq \bigoplus_{i \in I} \operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(\Sigma, \bar{F}(Y_i)) \simeq \bigoplus_{i \in I} \operatorname{Hom}_{\bar{\mathcal{A}}/\bar{\mathcal{C}}}(\bar{U}(\Sigma), Y_i) . \end{aligned}$$

Therefore, $\bar{U}(\Sigma)$ is a small object. □

Corollary 4.6. *If $\Sigma \in \mathcal{A}/\mathcal{C}$ is a simple object, then $\bar{U}(\Sigma)$ is a small object in $\bar{\mathcal{A}}/\bar{\mathcal{C}}$.* □

Proposition 4.7. *If Σ is a projective object in $\mathcal{A}/\mathcal{C}$, then $\bar{U}(\Sigma)$ is a projective object in $\bar{\mathcal{A}}/\bar{\mathcal{C}}$.*

Proof. Apply the dual of Proposition IV.9.5 of [14]. □

Proposition 4.8. *Let $M \in \mathcal{A}$ be an object such that the family $\{T_1(M(\sigma)) \mid \sigma \in G\}$ is a family of generators of the category $\mathcal{A}/\mathcal{C}$. If we put $\Sigma = T_1(M)$, then $\bar{U}(\Sigma)$ is a generator of the category $\bar{\mathcal{A}}/\bar{\mathcal{C}}$.*

Proof. Let $Y \in \bar{\mathcal{A}}/\bar{\mathcal{C}}$ be an arbitrary object of the category $\bar{\mathcal{A}}/\bar{\mathcal{C}}$. Then $S_2(Y) \in \bar{\mathcal{A}}$ and therefore there exists $X \in \mathcal{A}$ such that $S_2(Y)$ is a homomorphic image of X. Since $\{T_1(M(\sigma)) \mid \sigma \in G\}$ is a family of generators of the category $\mathcal{A}/\mathcal{C}$, then there exists an epimorphism in the category $\mathcal{A}/\mathcal{C}$

$$\bigoplus_{i \in I} T_1(M(\sigma_i)) \xrightarrow{\alpha} T_1(X) \to 0,$$

where $(\sigma_i)_{i \in I}$ is a family of elements of G. We have the exact sequence

$$S_1\Big(\bigoplus_{i \in I} T_1(M(\sigma_i))\Big) \xrightarrow{S_1(\alpha)} S_1T_1(X) \to \operatorname{Coker} S_1(\alpha) \to 0,$$

where $\operatorname{Coker} S_1(\alpha) \in \mathcal{C}$.

We have the canonical monomorphism

$$\beta : \bigoplus_{i \in I} S_1T_1(M(\sigma_i)) \to S_1\Big(\bigoplus_{i \in I} T_1(M(\sigma_i))\Big)$$

such that $\operatorname{Coker} \beta \in \mathcal{C}$. Since $\mathcal{C}$ is rigid, then

$$S_1T_1(M(\sigma_i)) \simeq (S_1T_1)(M)(\sigma_i),$$

hence we have the monomorphism

$$w : \bigoplus_{i \in I} (S_1T_1)(M)(\sigma_i) \to S_1\Big(\bigoplus_{i \in I} T_1(M(\sigma_i))\Big)$$

such that $\operatorname{Coker} w \in \mathcal{C}$.

Since $\mathcal{C} \subseteq \bar{\mathcal{C}}$ and T_2 commutes with direct sums, we have the canonical epimorphism

$$\bigoplus_{i \in I} (T_2U)(S_1T_1(M)(\sigma_i)) \to T_2US_1T_1(X) \to 0.$$

But $U(S_1T_1(M)(\sigma_i)) = US_1T_1(M)$; so we have an epimorphism

$$(T_2 \circ U \circ S_1 \circ T_1)(M)^{(I)} \to (T_2 \circ U \circ S_1 \circ T_1)(X) \to 0 .$$

We have the canonical morphism

$$X \xrightarrow{\psi_1(X)} (S_1T_1)(X)$$

where $\operatorname{Ker}\psi_1(X), \operatorname{Coker}\psi_1(X) \in \mathscr{C}$. Since $\mathscr{C} \subseteq \bar{\mathscr{C}}$, it follows that $(T_2 \circ U \circ S_1 \circ T_1)(X) \simeq T_2(U(X)) = T_2(X)$. On the other hand, Y is a homomorphic image of $T_2(X)$, since $T_2 \circ S_2(Y) \simeq Y$; and $T_2 \circ U \circ S_1 = \bar{U}$. Therefore there exists an epimorphism

$$(\bar{U}(T_1(M)))^{(I)} \to Y \to 0 .$$

Thus $(\bar{U}(\Sigma))^{(I)} \to Y \to 0$ and $\bar{U}(\Sigma)$ is a generator of the category $\bar{\mathscr{A}}/\bar{\mathscr{C}}$. □

5. Relative graded Clifford theory

Let $\mathscr{A}$ be a Grothendieck category and $\mathscr{C}$ a localizing subcategory of $\mathscr{A}$. A non-zero object $M \in \mathscr{A}$ is called *$\mathscr{C}$-simple* (or *$\mathscr{C}$-cocritical*) if (1) M is $\mathscr{C}$-torsionfree and (2) for any non-zero subobject $M' \neq 0$ of M, we have $M/M' \in \mathscr{C}$. If $T : \mathscr{A} \to \mathscr{A}/\mathscr{C}$ is the canonical functor, then M *is* $\mathscr{C}$-simple if and only if M is $\mathscr{C}$-torsionfree and $T(M)$ is a simple object in the category $\mathscr{A}/\mathscr{C}$. As in [15], an object $M \in \mathscr{A}$ is called *$\mathscr{C}$-semicocritical* if there exists a finite set $K_1, K_2, \ldots, K_n$ of subobjects of M such that $\bigcap_{i=1}^{n} K_i = 0$ and M/K_i is $\mathscr{C}$-simple ($\mathscr{C}$-cocritical) for each $i = 1, \ldots, n$. A category $\mathscr{A}$ is called *semisimple* if every object of $\mathscr{A}$ is semi-simple (i.e. every object of $\mathscr{A}$ is a direct sum of simple objects). By Proposition 0.2 of [10], if M is $\mathscr{C}$-torsionfree, then M is $\mathscr{C}$-semicocritical if and only if $T(M)$ is a semi-simple object of finite length in the category $\mathscr{A}/\mathscr{C}$.

If $M \in \mathscr{A}$, we can consider the closed subcategory $\sigma[M]$. Then if $\mathscr{C}$ is a localizing subcategory of $\mathscr{A}$, $\mathscr{C} \cap \sigma[M]$ is a localizing subcategory of $\sigma[M]$.

Proposition 5.1. *The following assertions hold with the above notation*:

(i) *M is $\mathscr{C}$-simple if and only if M is $\mathscr{C} \cap \sigma[M]$-simple, considered as an object in the category $\sigma[M]$.*

(ii) *If $M = \bigoplus_{i \in I} M_i$ and if each M_i, $i \in I$ is $\mathscr{C}$-simple, then the quotient category $\sigma[M]/\mathscr{C} \cap \sigma[M]$ is semisimple.*

Proof. Assertion (i) is obvious; so we prove only (ii). Let $T : \sigma[M] \to \sigma[M]/\mathscr{C} \cap \sigma[M]$ be the canonical functor. If $Y \in \sigma[M]/\mathscr{C} \cap \sigma[M]$, then there exists $N \in \sigma[M]$ such that $Y \simeq T(N)$. But then there exists an object $P \in \mathscr{A}$ and an epimorphism $M^{(I)} \xrightarrow{u} P \to 0$ for some set I, such that N is a subobject of P. Since T is exact and commutes with direct sums, we have the exact sequence

$$T(M)^{(I)} \xrightarrow{T(u)} T(P) \to 0 .$$

By (i), $T(M_i)$ is a simple object in the quotient category $\sigma[M]/\mathscr{C} \cap \sigma[M]$ for any $i \in I$. Therefore $T(P)$ is semi-simple and so $Y = T(N)$ is also semisimple. Hence $\sigma[M]/\mathscr{C} \cap \sigma[M]$ is a semisimple category. $\square$

Remark. By the proof of Proposition 5.1, the family $\{T(M_i) \mid i \in I\}$ is a family of generators in the category $\sigma[M]/\mathscr{C} \cap \sigma[M]$.

If R is a G-graded ring, we consider the category $\mathscr{A} = R$-**gr**. Let $\mathscr{C}$ be a rigid localizing subcategory of R-**gr** and $M \in R$-**gr**. If M is $\mathscr{C}$-simple (or $\mathscr{C}$-cocritical), then M is called a gr-$\mathscr{C}$*-simple module*.

Let $\bar{\mathscr{C}}$ be the smallest localizing subcategory of R-**mod** containing $\mathscr{C}$. We consider the category $\sigma_R[M]$, where M is considered as an object in R-**mod**, i.e. $\sigma_R[M]$ is the class of all R-modules subgenerated by M. Since $\bar{\mathscr{C}} \cap \sigma_R[M]$ is a localizing subcategory of $\sigma_R[M]$, we denote by $T : \sigma_R[M] \to \sigma_R[M]/\bar{\mathscr{C}} \cap \sigma_R[M]$ the canonical functor. We denote by $\Sigma = T(M)$ and by $\Delta = \mathrm{End}(\Sigma)$ the ring of endomorphisms of the object Σ.

With the above notation, we are now in a position to give the main result of this paper.

Theorem 5.2. (Relative Clifford Theorem) *Assume that M is a gr-$\mathscr{C}$-simple module. Then Σ is a small projective generator in the quotient category $\sigma_R[M]/\bar{\mathscr{C}} \cap \sigma_R[M]$. In particular, it follows that the functor*

$$\mathrm{Hom}_{\sigma_R[M]/\bar{\mathscr{C}} \cap \sigma_R[M]}(\Sigma, -) : \sigma_R[M]/\bar{\mathscr{C}} \cap \sigma_R[M] \to \Delta\text{-}\mathbf{mod}$$

is an equivalence of categories.

Proof. We denote by $\sigma^{\mathrm{gr}}[M] = \sigma_{R\text{-}\mathbf{gr}}[\bigoplus_{\sigma \in G} M(\sigma)]$. Since $\bigoplus_{\sigma \in G} M(\sigma)$ is a G-invariant graded R-module, then $\sigma^{\mathrm{gr}}[M]$ is a rigid closed subcategory of R-**gr**. Obviously, we have $\overline{\sigma^{\mathrm{gr}}[M]} = \sigma_R[M]$. Since M is gr-$\mathscr{C}$-simple and $\mathscr{C}$ is a rigid localizing subcategory of R-**gr**, then $M(\sigma)$ is gr-$\mathscr{C}$-simple for any $\sigma \in G$. By Proposition 5.1 we get that $\sigma^{\mathrm{gr}}[M]/\mathscr{C} \cap \sigma^{\mathrm{gr}}[M]$ is a semi-simple category.

We prove now the equality $\overline{\mathscr{C} \cap \sigma^{\mathrm{gr}}[M]} = \bar{\mathscr{C}} \cap \sigma_R[M]$. Since $\mathscr{C} \cap \sigma^{\mathrm{gr}}[M] \subseteq \bar{\mathscr{C}} \cap \sigma_R[M]$, then $\overline{\mathscr{C} \cap \sigma^{\mathrm{gr}}[M]} \subseteq \bar{\mathscr{C}} \cap \sigma_R[M]$. Conversely, if $N \in \bar{\mathscr{C}} \cap \sigma_R[M]$, then $N \in \bar{\mathscr{C}}$ and by Proposition 4.2, $F(N) \in \mathscr{C}$. Since $F(N) = \bigoplus_{\sigma \in G} N(\sigma)$ and F is an exact functor, it is clear that $F(N) \in \sigma^{\mathrm{gr}}[M]$. Hence $F(N) \in \mathscr{C} \cap \sigma^{\mathrm{gr}}[M]$. By Proposition 4.2, we obtain that $N \in \overline{\mathscr{C} \cap \sigma^{\mathrm{gr}}[M]}$.

We denote by

$$T^{gr} : \sigma^{gr}[M] \to \sigma^{gr}[M]/\mathscr{C} \cap \sigma^{gr}[M]$$

the canonical functor. If S (resp. S^{gr}) denotes the right adjoint of T (resp. T^{gr}), we denote by $\bar{U}$ and $\bar{F}$ the functors

$$\bar{U} = T \circ U \circ S^{gr} : \sigma^{gr}[M]/\mathscr{C} \cap \sigma^{gr}[M] \to \sigma_R[M]/\bar{\mathscr{C}} \cap \sigma_R[M]$$

and

$$\bar{F} = T^{gr} \circ F \circ S : \sigma_R[M]/\bar{\mathscr{C}} \cap \sigma_R[M] \to \sigma^{gr}[M]/\mathscr{C} \cap \sigma^{gr}[M] .$$

By the Remark after Proposition 5.1, the family $\{T^{gr}(M(\sigma)) \mid \sigma \in G\}$ is a family of generators in the quotient category $\sigma^{gr}[M]/\mathscr{C} \cap \sigma^{gr}[M]$. By Proposition 4.8 we get that $\bar{U}(T^{gr}(M))$ is a generator in the category $\sigma_R[M]/\bar{\mathscr{C}} \cap \sigma_R[M]$. But $(\bar{U} \circ T^{gr})(M) = (T \circ U \circ S^{gr} \circ T^{gr})(M)$. We have the exact sequence

$$0 \to \operatorname{Ker} \psi^{gr}(M) \to M \xrightarrow{\psi^{gr}(M)} (S^{gr} \circ T^{gr})(M) \to \operatorname{Coker} \psi^{gr}(M) \to 0 ,$$

where $\operatorname{Ker} \psi^{gr}(M)$,$\operatorname{Coker} \psi^{gr}(M) \in \mathscr{C} \subseteq \bar{\mathscr{C}}$. Hence $T \circ U(\psi^{gr}(M))$ is an isomorphism. Therefore, $\bar{U}(T^{gr}(M)) \simeq T(M)$. Hence Σ is a generator in the category $\sigma_R[M]/\bar{\mathscr{C}} \cap \sigma_R[M]$. On the other hand, by Propositions 4.5, 4.6 and 4.7 it follows that Σ is a small projective generator. Thus, by Mitchell's Theorem (see [8, Theorem 4.1, p. 104]) it follows that the category $\sigma_R[M]/\bar{\mathscr{C}} \cap \sigma_R[M]$ is equivalent to Δ-**mod** via the functor

$$\operatorname{Hom}_{\sigma_R[M]/\bar{\mathscr{C}} \cap \sigma_R[M]}(\Sigma, -) : \sigma_R[M]/\bar{\mathscr{C}} \cap \sigma_R[M] \to \Delta\text{-}\mathbf{mod} . \qquad \square$$

Remark. If in Theorem 5.2 we put $\mathscr{C} = \{0\}$, then M is $\mathscr{C}$-simple if and only if M is gr-simple (i.e. simple object in R-**gr**). In this case we obtain that $\sigma_R[M]$ is equivalent with the category Δ-**mod**, where $\Delta = \operatorname{End}_R(M)$, via the functor

$$\operatorname{Hom}_R(M, -) : \sigma_R[M] \to \Delta\text{-}\mathbf{mod} .$$

This particular case of Theorem 5.2 is exactly the graded Clifford Theorem in [2], [3] and [5].

For the applications of Theorem 5.2 it will be very important to study the ring $\Delta = \operatorname{End}_{\sigma_R[M]/\bar{\mathscr{C}} \cap \sigma_R[M]}(\Sigma)$, where $\Sigma = T(U(M))$. We have the exact sequence

$$0 \to \operatorname{Ker} \psi^{gr}(M) \to M \xrightarrow{\psi^{gr}(M)} (S^{gr} \circ T^{gr})(M) \to \operatorname{Coker} \psi^{gr}(M) \to 0 ,$$

where $\operatorname{Ker} \psi^{gr}(M)$,$\operatorname{Coker} \psi^{gr}(M) \in \mathscr{C} \cap \sigma^{gr}[M]$. Hence $\operatorname{Ker} \psi^{gr}(M)$, $\operatorname{Coker} \psi^{gr}(M) \in \bar{\mathscr{C}} \cap \sigma_R[M]$, and therefore $\Sigma = (T \circ U)((S^{gr} \circ T^{gr})(M))$. Hence we can assume that $M \simeq (S^{gr} \circ T^{gr})(M)$, i.e. M is $\mathscr{C} \cap \sigma^{gr}[M]$-closed.

We denote by $H = G\{M\}$ the stabilizer of M, i.e.

$$H = \{\sigma \in G \mid M(\sigma) \simeq M\}\ .$$

Since $\mathscr{C} \cap \sigma^{\mathrm{gr}}[M]$ is a rigid localizing subcategory of $\sigma^{\mathrm{gr}}[M]$, then

$$H = \{\sigma \in G \mid T^{\mathrm{gr}}(M(\sigma)) \simeq T^{\mathrm{gr}}(M)\}\ .$$

We consider the G-graded ring $\Delta = \mathrm{END}_R(M)$. This ring is a subring of $\mathrm{End}_R(M)$, i.e. the ring of all R-endomorphisms of M. We have the following result.

Proposition 5.3. *With the above hypotheses, we have the following assertions*:

(1) $\mathrm{END}_R(M) = \mathrm{End}_R(M)$.

(2) *For any* $\sigma \not\in H$, $\mathrm{END}_R(M)_\sigma = 0$.

(3) *If* $\sigma \in H$, *then every non-zero element of* $\mathrm{END}_R(M)$ *is an invertible element in the ring* $\mathrm{END}_R(M)$. *In particular*, $\mathrm{END}_R(M)_1$ *is a division ring.*

(4) *We have* $\mathrm{End}_R(M) = \bigoplus_{\sigma\in H} \mathrm{END}_R(M)_\sigma$ *and* $\mathrm{End}_R(M)$ *is an* H*-crossed product.*

Proof. (1) Let $f \in \mathrm{End}_R(M)$. We consider a non-zero finitely generated graded submodule M' of M. Since $T^{\mathrm{gr}}(M)$ is a simple object in the quotient category $\sigma^{\mathrm{gr}}[M]/\mathscr{C} \cap \sigma^{\mathrm{gr}}[M]$, then $M/M' \in \mathscr{C} \cap \sigma^{\mathrm{gr}}[M]$.

Let $g = f|_{M'}$. Since M' is finitely generated, then $g \in \mathrm{HOM}_R(M', M)$, i.e. $g = g_1 + \cdots + g_s$, where $g_i : M' \to M$ is an R-morphism of degree $\sigma_i \in G$. Now by the hypothesis M is $\mathscr{C} \cap \sigma^{\mathrm{gr}}[M]$-closed in the category $\sigma^{\mathrm{gr}}[M]$. We consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "g_i"'] & M \arrow[r, "\pi"] \arrow[dl, "h_i"] & M/M' \arrow[r] & 0 \\ & M(\sigma_i) & & & \end{tikzcd}$$

where $M/M' \in \mathscr{C} \cap \sigma^{\mathrm{gr}}[M]$ and $g_i \in \mathrm{Hom}_{R\text{-}\mathbf{gr}}(M', N(\sigma_i))$. Since $M(\sigma_i)$ remains $\mathscr{C} \cap \sigma^{\mathrm{gr}}[M]$-closed, then there exists $h_i \in \mathrm{Hom}_{R\text{-}\mathbf{gr}}(M, M(\sigma_i))$ such that $h_i|_{M'} = g_i$. If we put $h = h_1 + \cdots + h_s$, then $h \in \mathrm{END}_R(M)$ and $(f - g)(M') = 0$. Then there exists a morphism $u : M/M' \to M$ such that $f - h = u \circ \pi$. But $M/M' \in \mathscr{C} \cap \sigma^{\mathrm{gr}}[M]$, then $M/M' \in \mathscr{C} \cap \sigma[M]$ and therefore $u(M/M') \in \mathscr{C} \cap \sigma[M]$. Since M is $\mathscr{C}$-torsionfree, then M is also $\bar{\mathscr{C}}$-torsionfree. Thus $u(M/M') = 0$ and therefore $f - g = 0$; so $f = g$. Hence $f \in \mathrm{END}_R(M)$.

(2) Let $\sigma \in G$, $\sigma \not\in H$. Assume $f \in \mathrm{END}_R(M)_\sigma$, $f \neq 0$. Hence $f \in \mathrm{Hom}_{R\text{-}\mathbf{gr}}(M, M(\sigma))$. Therefore, $T^{\mathrm{gr}}(f) : T^{\mathrm{gr}}(M) \to T^{\mathrm{gr}}(M(\sigma))$ and $T^{\mathrm{gr}}(f) \neq 0$ because $M(\sigma)$ is $\mathscr{C}$-torsionfree. Since $T^{\mathrm{gr}}(M)$ and $T^{\mathrm{gr}}(M(\sigma))$ are simple objects, it

follows that T^{gr} is an isomorphism. Hence $T^{gr}(M) \simeq T^{gr}(M(\sigma))$, and $\sigma \not\in H$, which is a contradiction.

(3) Assume now that $\sigma \in H$ and let $f \in \mathrm{END}_R(M)_\sigma$, and $f \neq 0$. Then $f \in \mathrm{Hom}_{R\text{-gr}}(M, M(\sigma))$. If $T^{gr}(f) : T^{gr}(M) \to T^{gr}(M(\sigma))$ is the zero morphism, then $\mathrm{Im}\, f \in \mathscr{C} \cap \sigma^{gr}[M]$. Since $M(\sigma)$ is $\mathscr{C} \cap \sigma^{gr}[M]$-torsionfree, then $\mathrm{Im}\, f = 0$. Hence $f = 0$, which is a contradiction.

Since $T^{gr}(M)$ and $T^{gr}(M(\sigma))$ are simple objects and $T^{gr}(f) \neq 0$, then $T^{gr}(f)$ is an isomorphism. Therefore $(S^{gr} \circ T^{gr})(f) : (S^{gr} \circ T^{gr})(M) \to (S^{gr} \circ T^{gr})(M(\sigma))$ is an isomorphism. Since by hypothesis $M \simeq (S^{gr} \circ T^{gr})(M)$, it follows that f is an isomorphism.

(4) This follows from (2) and (3). $\square$

Since we have the isomorphism

$$T \circ U \simeq \bar{U} \circ T^{gr} \quad \text{(Theorem 3.2)},$$

then we have that

$$\begin{aligned} \Delta &= \mathrm{End}_{\sigma[M]/\bar{\mathscr{C}} \cap \sigma[M]}(\Sigma) \\ &= \mathrm{Hom}_{\sigma[M]/\bar{\mathscr{C}} \cap \sigma[M]}((\bar{U} \circ T^{gr})(M), (\bar{U} \circ T^{gr})(M)) \\ &= \mathrm{Hom}_{\sigma^{gr}[M]/\mathscr{C} \cap \sigma^{gr}[M]}(T^{gr}(M), (\bar{F} \circ \bar{U})(T^{gr}(M))) . \end{aligned}$$

Since $\bar{F} \circ \bar{U} = T^{gr} \circ F \circ S \circ T \circ U \circ S^{gr}$ and since for the canonical morphism

$$M \xrightarrow{\psi^{gr}(M)} (S^{gr} \circ T^{gr})(M)$$

we have $\mathrm{Ker}\, \psi^{gr}(M)$,$\mathrm{Coker}\, \psi^{gr}(M) \in \mathscr{C} \cap \sigma^{gr}[M]$, then it follows that

$$(S \circ T \circ U \circ S^{gr} \circ T^{gr})(M) \simeq (S \circ T)(M) .$$

On the other hand, if we consider the exact sequence

$$0 \to \mathrm{Ker}\, \psi(M) \to M \xrightarrow{\psi(M)} (S \circ T)(M) \to \mathrm{Coker}\, \psi(M) \to 0$$

where $\mathrm{Ker}\, \psi(M)$,$\mathrm{Coker}\, \psi(M) \in \bar{\mathscr{C}} \cap \sigma[M]$, then we obtain the exact sequence

$$\begin{aligned} &0 \to F(\mathrm{Ker}\, \psi(M)) \to F(M) \xrightarrow{F(\psi(M))} (F \circ S \circ T)(M) \\ &\quad \to F(\mathrm{Coker}\, \psi(M)) \to 0 \end{aligned}$$

where $F(\mathrm{Ker}\, \psi(M))$, and $F(\mathrm{Coker}\, \psi(M))$ belong to $\mathscr{C} \cap \sigma^{gr}[M]$ (Proposition 4.2).

Thus we have

$$(\bar{F}\circ\bar{U})(T^{\mathrm{gr}}(M)) \simeq T^{\mathrm{gr}}(F(M)) = T^{\mathrm{gr}}\Big(\bigoplus_{\sigma\in G} M(\sigma)\Big) \simeq \bigoplus_{\sigma\in G} T^{\mathrm{gr}}(M(\sigma))\,.$$

Therefore, we have the canonical isomorphism of abelian groups

$$\begin{aligned}\Delta &\simeq \mathrm{Hom}_{\sigma^{\mathrm{gr}}[M]/\mathscr{C}\cap\sigma^{\mathrm{gr}}[M]}\Big(T^{\mathrm{gr}}(M), \bigoplus_{\sigma\in G} T^{\mathrm{gr}}(M(\sigma))\Big)\\ &\simeq \bigoplus_{\sigma\in G}\mathrm{Hom}_{\sigma^{\mathrm{gr}}[M]/\mathscr{C}\cap\sigma^{\mathrm{gr}}[M]}(T^{\mathrm{gr}}(M), T^{\mathrm{gr}}(M(\sigma)))\\ &\simeq \bigoplus_{\sigma\in G}\mathrm{Hom}_{R\text{-}\mathbf{gr}}((S^{\mathrm{gr}}\circ T^{\mathrm{gr}})(M), (S^{\mathrm{gr}}\circ T^{\mathrm{gr}})(M(\sigma)))\,.\end{aligned}$$

Since we assumed that $M \simeq (S^{\mathrm{gr}}\circ T^{\mathrm{gr}})(M)$, then we conclude that

$$\Delta \simeq \bigoplus_{\sigma\in G}\mathrm{Hom}_{R\text{-}\mathbf{gr}}(M, M(\sigma)) \tag{4}$$

(isomorphism of abelian groups).

Then we define the canonical morphism of rings

$$\varphi : \mathrm{End}_R(M) \to \Delta$$

by $\varphi(f) = T(f)$ for any $f \in \mathrm{End}_R(M)$.

With the above notation, we have the following result.

Theorem 5.4. *The canonical morphism*

$$\varphi : \mathrm{End}_R(M) \to \Delta\,,$$

$\varphi(f) = T(f)$ is an isomorphism of rings. If we put $\Delta_\sigma = \varphi(\mathrm{END}_R(M)_\sigma)$ for any $\sigma\in G$, then $\Delta = \bigoplus_{\sigma\in G}\Delta_\sigma$ and Δ is a H-crossed product.

Proof. If $\varphi(f) = 0$, where $f \in \mathrm{End}_R(M)$, then $T(f) = 0$ and therefore $\mathrm{Im}\, f \in \mathscr{C}\cap\sigma[M]$. But M is $\mathscr{C}\cap\sigma[M]$-torsionfree; hence $\mathrm{Im}\, f = 0$. Then $f = 0$; so φ is an injective morphism.

From the isomorphism (4) it follows that φ is also surjective. The rest of the theorem follows from Proposition 5.3. □

6. Applications

Let $R = \bigoplus_{\sigma\in G} R_\sigma$ be a G-graded ring and let $\mathscr{C}$ be a rigid localizing subcategory of R-**gr**. If $M \in R$-**gr**, we denote by $\sigma^{\mathrm{gr}}[M]$ the smallest closed subcategory of R-**gr** containing M. We consider the canonical functors

$$T^{\mathrm{gr}} : \sigma^{\mathrm{gr}}[M] \to \sigma^{\mathrm{gr}}[M]/\mathscr{C} \cap \sigma^{\mathrm{gr}}[M]$$

and

$$S^{\mathrm{gr}} : \sigma^{\mathrm{gr}}[M]/\mathscr{C} \cap \sigma^{\mathrm{gr}}[M] \to \sigma^{\mathrm{gr}}[M]$$

the right adjoint of T^{gr}. We denote by

$$H = \{\sigma \in G \mid T^{\mathrm{gr}}(M) \simeq T^{\mathrm{gr}}(M(\sigma))\} .$$

With the above notation we have the following result.

Theorem 6.1. (Relative Maschke's Theorem) *Assume that M is $\mathscr{C}$-simple (or $\mathscr{C}$-cocritical) and the subgroup H is finite. If we put $n = |H|$, then either $nM = 0$ or M is $\bar{\mathscr{C}}$-semicocritical.*

Proof. We use the notions of Section 5. By the *Relative Clifford Theorem* we have that the quotient category $\sigma[M]/\bar{\mathscr{C}} \cap \sigma[M]$ is equivalent to the category Δ-**mod**, where $\Delta = \mathrm{End}_{\sigma[M]/\bar{\mathscr{C}} \cap \sigma[M]}(\Sigma)$ and $\Sigma = (T \circ U)(M)$. Assume now that $nM \neq 0$. Since M is $\mathscr{C}$-cocritical, then we have the exact sequence

$$0 \to M \xrightarrow{\alpha_n} M \to M/nM \to 0 ,$$

where $\alpha_n(x) = nx$, $x \in M$ and $M/nM \in \mathscr{C}$. Then it follows that $T^{\mathrm{gr}}(\alpha_n)$ is an isomorphism. Since we have the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\alpha_n} & M \\ \downarrow{\scriptstyle \psi^{\mathrm{gr}}(M)} & & \downarrow{\scriptstyle \psi^{\mathrm{gr}}(M)} \\ (S^{\mathrm{gr}} \circ T^{\mathrm{gr}})(M) & \xrightarrow{\beta_n} & (S^{\mathrm{gr}} \circ T^{\mathrm{gr}})(M) \end{array}$$

where $\beta_n(y) = ny$ for any $y \in (S^{\mathrm{gr}} \circ T^{\mathrm{gr}})(M)$. Then it follows that $\beta_n = (S^{\mathrm{gr}} \circ T^{\mathrm{gr}})(\alpha_n)$ and therefore n is invertible on $N = (S^{\mathrm{gr}} \circ T^{\mathrm{gr}})(M)$. By Theorem 5.4, $\Delta = \mathrm{END}_R(N) = \bigoplus_{\sigma \in H} \mathrm{END}_R(N)_\sigma$. Since n is invertible in Δ, by the classical *Maschke's Theorem*, it follows that Δ is a semisimple artinian ring. By the *Relative Clifford Theorem*, the functor

$$\mathrm{Hom}(\Sigma, -) : \sigma[M]/\bar{\mathscr{C}} \cap \sigma[M] \to \Delta\text{-}\mathbf{mod}$$

is an equivalence of categories. Hence Σ is a semi-simple object of finite length. Therefore, M is $\bar{\mathscr{C}}$-semi-cocritical. □

Remarks. (1) In the paper [10] this result was proved with other methods for the case in which G is a finite group (see [10, Theorem 3.1]).

(2) When we take $\mathscr{C} = \{0\}$, then M is $\mathscr{C}$-simple if and only if M is a simple object in R-**gr**. In this case $H = G\{M\}$. Then, in this particular case we have the following result: if $n = |G\{M\}| < \infty$ and M is gr-simple, then either $nM = 0$ or M is semi-simple of finite length (this was proved in Theorem 3.2 of [5]).

Let $\mathscr{A}$ be a Grothendieck category. The Gabriel dimension of an object in $\mathscr{A}$ can be defined by using the Gabriel filtrations on $\mathscr{A}$ (see [7, p. 3]). One considers the localizing subcategories $\mathscr{C}_\alpha$ of $\mathscr{A}$ and the canonical functors

$$T_\alpha : \mathscr{A} \to \mathscr{A}/\mathscr{C}_\alpha$$

defined by transfinite recursion: $\mathscr{C}_0 = 0$, $T_0 =$ identity functor of $\mathscr{A}$. If α is not a limit ordinal, then $\mathscr{C}_\alpha$ is the smallest localizing subcategory of $\mathscr{A}$ that contains all objects $X \in \mathscr{A}$ such that $T_{\alpha-1}(X)$ has finite length. If α is a limit ordinal, then $\mathscr{C}_\alpha$ is the smallest localizing subcategory of $\mathscr{A}$, that contains $\bigcup_{\beta<\alpha} \mathscr{C}_\beta$. When an object X belongs to $\mathscr{C}_\alpha$, then we say that X has Gabriel dimension and G-dim X is the least such ordinal.

If α is not a limit ordinal, then an object $X \in \mathscr{A}$, which has the property that X is $\mathscr{C}_{\alpha-1}$-simple, is called α-simple. In particular X is 1-simple if and only if X is a simple object in $\mathscr{A}$.

If R is a G-graded ring, we can consider the cases when $\mathscr{A}$ is R-**gr** or R-**mod**. We denote by $\mathscr{C}_\alpha^{\mathrm{gr}}$ (resp. $\mathscr{C}_\alpha$) the Gabriel filtration associated to the category R-**gr** (resp. R-**mod**). If $M \in R$-**gr**, then we denote by gr-G-dim M (resp. G-dim M) the Gabriel dimension of M in R-**gr** (resp. in R-**mod**) in the case that this dimension exists. Then we have

$$\mathscr{C}_\alpha^{\mathrm{gr}} = \{M \in R - \mathbf{gr} \mid \text{gr-}G\text{-dim } M \le \alpha\}$$

(resp.

$$\mathscr{C}_\alpha = \{N \in R\text{-}\mathbf{mod} \mid \text{G-dim } N \le \alpha\}) .$$

We obtained that $\mathscr{C}_\alpha^{\mathrm{gr}}$ (resp. $\mathscr{C}_\alpha$) is a rigid localizing subcategory of R-**gr** (resp. is a localizing subcategory of R-**mod**) for all ordinals α.

We recall that a group G is called polycyclic-by-finite if there exists a finite subnormal series

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

such that each factor G_{i+1}/G_i is a finite or infinite cyclic group. The number of finite cyclic factors from this series is called the Hirch number of G and is denoted by $h(G)$ (it does not depend on the particular series chosen).

In connection with Gabriel dimension, if $X \in \mathscr{A}$ is an arbitrary object in the Grothendieck category $\mathscr{A}$, we can define the *Krull dimension* of X, which will be denoted by K-dim X (see [6, 12]). This is defined by transfinite recursion as follows: if $X = 0$, K-dim $X = \alpha$ provided that there is no infinite descending chain

$$X = X_0 \supset X_1 \supset \cdots$$

of subobjects X_i such that, for $i = 1, 2, \ldots$, K-dim$(X_{i-1}/X_i) \nless \alpha$. It is possible that there is no ordinal α such that K-dim $X = \alpha$. In this case we say X has no Krull dimension. If $X \in \mathscr{A}$ has the Krull dimension, then it is well known that X has the Gabriel dimension and G-dim $X \leq$ K-dim $X + 1$ (Corollary 2.2 of [7]). When X is a noetherian object, then G-dim $X =$ K-dim $X + 1$ (Proposition 2.3 of [7]).

If $X \in \mathscr{A}$ and α is an ordinal, then X is called α-critical if K-dim $X = \alpha$ and K-dim $X' < \alpha$ for each proper homomorphic image X' of X.

If $\mathscr{A} = R$-**gr** or $\mathscr{A} = R$-**mod** and $M \in R$-**gr** we denote by gr-K-dim M (resp. K-dim M) the Krull dimension of M in R-**gr** (resp. R-**mod**) in the case this dimension exists.

If $R = \bigoplus_{\sigma \in G} R_\sigma$ is a G-graded ring, where G is a polycyclic-by-finite group and $M \in R$-**gr** is gr-noetherian, then M is noetherian in R-**mod** and moreover gr-K-dim $M \leq$ K-dim $M \leq$ gr-K-dim $M + h(G)$ (see [1]).

We are now in position to state and prove the main result of this section.

Theorem 6.2. *Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a G-graded ring, where G is a polycyclic-by-finite group. Let $M \in R$-**gr** such that* gr-G-dim $M = \varepsilon + n$, *where $\varepsilon = 0$ or ε is a limit ordinal and $n \in \mathcal{N}$. Then* G-dim M *exists and we have the inequality*

$$\text{G-dim } M \leq \varepsilon + n(1 + h(G)) .$$

Proof. We show the result by transfinite recursion on $\alpha = \varepsilon + n$.

If $\alpha = 1$, then gr-G-dim $M = 1$, and we reduce the proof (using the Loewy series) to the case when M is gr-simple. But in this case, from the above result, we have that M is noetherian in R-**mod** with K-dim $M \leq h(G)$. Since G-dim $M \leq$ K-dim $M + 1$, then G-dim $M \leq 1 + 1h(G)$.

We suppose now that the assertion is true for all ordinals $\beta < \alpha = \varepsilon + n$, and prove it for α. If $\alpha = \varepsilon + n$, $n \neq 0$, then we can reduce the problem to the case in which M is gr-α-simple, so that M is $\overline{\mathscr{C}^{\text{gr}}_{\alpha-1}}$-simple. For simplifying, we let $\mathscr{C} = \overline{\mathscr{C}^{\text{gr}}_{\alpha-1}}$. If $T : \sigma_R[M] \to \sigma_R[M]/\mathscr{C} \cap \sigma_R[M]$ is the canonical functor, then by Theorem 5.2 we have that $\sigma_R[M]/\mathscr{C} \cap \sigma_R[M]$ is equivalent to the category Δ-**mod**, where $\Delta = \text{End}_{\sigma_R[M]/\mathscr{C} \cap \sigma_R[M]}(T(M))$. By Theorem 5.4 Δ is a H-crossed product, where H is subgroup of G. Since G is a polycyclic-by-finite group, then Δ is a (left and right) noetherian ring. Then $T(M)$ is a noetherian object in the category $\sigma_R[M]/\mathscr{C} \cap \sigma_R[M]$ with the relative Krull dimension less than $h(G)$. Now by [9, Proposition 1.3], we have that M has Gabriel dimension on R-**mod** and G-dim $M \leq \varepsilon + (n-1)(1 + h(G)) + h(G) + 1 = \varepsilon + n(1 + h(G))$.

Now if $\alpha = \varepsilon$ is a limit ordinal, then $M \in \mathscr{C}^{gr}_{\alpha+1}$, where $\mathscr{C}^{gr}_{\alpha+1}$ is the smallest localizing subcategory of R-**gr** containing $\bigcup_{\beta<\alpha} \mathscr{C}^{gr}_{\beta}$. But by the induction hypothesis, $\overline{\mathscr{C}^{gr}_{\beta}} \subseteq \mathscr{C}_{\alpha}$ (α is limit ordinal). Hence $\mathscr{C}^{gr}_{\alpha} \subseteq \mathscr{C}_{\alpha}$ and therefore M has the Gabriel dimension in R-**mod** and G-**dim** $M \leq \alpha$. □

Remark. This result generalizes the main results from [9].

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